arXiv:nlin/0001035v1 [nlin.CD] 18 Jan 2000

Lévy Anomalous Diffusion and Fractional Fokker–Planck Equation

V.V. Yanovsky, A.V. Chechkin
Turbulence Research
Institute for Single Crystals
National Acad. Sci. Ukraine,
Lenin ave. 60, Kharkov 310001, Ukraine

D. Schertzer
Laboratoire de Modélisation en Mécanique, Tour 66, Boite 162
Université Pierre et Marie Curie,
4 Place Jussieu F-75252 Paris Cedex 05, France.

A.V. Tur(Tour)
Centre d'Etude Spatiale des Rayonnements
9 Avenue Colonel-Roche,BP 4346, 31028 Toulouse Cedex 4, France

(Submitted to Physica A, 12 June 1997, revised version: 22 January 1999, accepted: December 1999)

Abstract

We demonstrate that the Fokker-Planck equation can be generalized into a 'Fractional Fokker-Planck' equation, i.e. an equation which includes fractional space differentiations, in order to encompass the wide class of anomalous diffusions due to a Lévy stable stochastic forcing. A precise determination of this equation is obtained by substituting a Lévy stable source to the classical gaussian one in the Langevin equation. This yields not only the anomalous diffusion coefficient, but a non trivial fractional operator which corresponds to the possible asymmetry of the Lévy stable source. Both of them cannot be obtained by scaling arguments. The (mono-) scaling behaviors of the Fractional Fokker-Planck equation and of its solutions are analysed and a generalization of the Einstein relation for the anomalous diffusion coefficient is obtained.

This generalization yields a straightforward physical interpretation of the parameters of Lévy stable distributions. Furthermore, with the help of important examples, we show the applicability of the Fractional Fokker-Planck equation in physics.

1 Introduction

The Fokker–Planck equation is one of the classical, widely used equations of statistical physics. It describes a broad spectrum of problems related to the evolution of various dynamic systems under the influence of stochastic forces and has numerous applications, see, e.g. [1]. Usually, the Fokker–Planck equation can be derived following the Langevin approach, that is, starting from the stochastic "equation of motion" for the dynamic variable whose probability distribution we are interested. In this approach, the basic assumptions on the "random force/source" in this equation of motion are usually that they have: (i) Gaussian statistics, and (ii) delta-correlated correlation. These two assumptions yield the (classical) Fokker–Planck equation.

These assumptions are physically motivated by the fact that the random source is a sum of a large number of independent identical random "pulses". If these quantities possess a finite variance, then, according to the Central Limit Theorem, the distribution of their sum tends to the normal law when the number of pulses go to infinity.

However, the Central Limit Theorem can be generalized for independent identically distributed (i.i.d.) random variables having non finite variance. Indeed, Lévy and Khintchine [2, 3, 4, 5] discovered a broader class of stable distributions. They correspond to the limit of normalized sums of i.i.d. stochastic variables. Each stable law has a characteristic index α $(0 < \alpha \leq 2)$, often called the Lévy stability index or the Lévy index, which is the critical order for the convergence of statistical moments. Indeed, a statistical moment of a given stable law is finite only if its order μ is strictly smaller than its Lévy index α (i.e. $\mu < \alpha$). Every moment of order higher order (including: $\mu = \alpha$) are infinite or, as often said, divergent. The only exception is the normal distribution which corresponds to the particular stable law which has its Lévy's index $\alpha = 2$ and the exceptional property that all its moments are finite.

The classical and rather academic example of the application of the Lévy stable laws is the Holtzmark distribution [6, 7] which is the distribution function of the gravitational force created at a randomly chosen point by a given system of stars. It is assumed that the system of stars is a (statistically) homogeneous set of physical points which mutually interact according to the gravitation law. Using dimensional consideration it can be shown [4] with a rather straightforward scaling argument that this distribution corresponds to a stable law with index $\alpha = 3/2$. We will see (Sect.7) that the following developments allow to generalize broadly this result.

Some other examples of application in physics of stable distributions can be found in review papers [8, 9] and in references therein.

However, let us recall that presumably the most well known application of Lévy stable laws in physics corresponds to the anomalous diffusion associated with a Lévy motion, also often called a 'Lévy flight' [10]. Indeed, one expects that with a Lévy stable forcing the cloud of particles will spread much faster

(for large times: $t >> 1$) than for a brownian motion. More precisely, we will confirm that the radius $r(t)$ of the cloud, at time t, has the following scaling law:

$$r(t) \sim t^{1/\alpha} \tag{1}$$

the lower bound being reached for the normal diffusion ($\alpha = 2$), Figs.1 and 2 display illustrations for comparison.

This scaling relation (Eq.1) is obviously incompatible with the Fokker-Planck equation, unless one substitutes a formal $(\alpha/2) - th$ power of the Laplacian to the classical Laplacian, therefore considers a Fractional Fokker-Planck equation, as suggested by [11]. Several authors, following different approaches, considered generalization of the Fokker-Planck equation in order to encompass the Lévy anomalous diffusion. On the one hand, particular cases of the Fractional Fokker-Planck equation were obtained [12, 13, 14]. However, this is not the only way to generalize the Fokker-Planck equation in order to respect the anomalous scaling relation of (Eq.1. Indeed, based on Tsallis[18]'s generalization of statistical mechanics, a nonlinear Fokker-Planck equation [15, 16] has been introduced and it was demonstrated [16, 17] that its solution respects also Eq.1. These approaches will be discussed and compared to ours in Sect. 6.

However, the interest in Lévy laws is not limited to anomalous diffusion. For instance, the rather large subclass of extremal $1/f$ or "pink" Lévy noises have been attracting much attention in the framework of multifractal fields. Indeed, it was shown that they correspond [19] to the attractive generators of "universal multifractals", which are the limit processes, under rather general conditions, of nonlinearly interacting i.i.d. multifractal processes. It is worthwhile to note that different techniques have been developed to simulate [20, 21] or analyse [22, 23] multifractal fields within the framework of Universal multifractals. Let us emphasize that the rather straightforward "Double Trace Moment" technique yields rather directly an estimate of the Lévy index α of the generator.

Furthermore, in most recent developments of multifractal studies, in particular those related to predictability of multifractal processes [24] , one needs having a kinetic equation for the generator, because the orientation of time axis becomes essential, contrary to earlier simulations, where the generator was obtained by isotropic (in time as well in space) fractional integration over a white noise.

As there is the need for a kinetic equation in the context of other examples/applications of Lévy laws in physics, this paper is devoted to establishing the corresponding "Fractional Fokker–Planck" equation.

In this paper we consider the time evolution of a stochastic variable forced by a random source having a stable distribution, i.e. a generalized Langevin equation (Sect. 2.1). We derive the corresponding kinetic equation for the distribution function of this stochastic variable, i.e. the Fractional Fokker–Planck equation which has fractional space derivative instead of the usual Laplacian

(Sect.2.2). We show that the expression of the Fractional Fokker–Planck equation is not unique (Sect.2.3) and determine its scale invariance group, as well as its scaling solutions (Sect.3). We also generalize the Einstein relation between the statistical exponents and the diffusion coefficient (Sect5). This helps us to clarify the physical meaning of the exponents characterizing a stable distribution. In Sect.6, we compare our approach to those followed by [12, 13, 14, 17]. Finally, we discuss (Sect.7) the possible applications of the theoretical results that we have obtained.

2 Fractional Fokker–Planck equation

2.1 Generalized Langevin equation

We start with the Langevin–like equation for a stochastic quantity $X(t)$:

$$\frac{dX(t)}{dt} = Y(t) \tag{2}$$

In the classical theory [1] of a Brownian motion, $X(t)$ is the location of Brownian particle under the influence of stochastic pulses $Y(t)$ [2]. The statistical properties of this stochastic forcing will be specified below. We first need to derive an equation for the distribution function

$$p(x,t) = \langle \delta[x - X(t)] \rangle \tag{3}$$

where the brackets $\langle ... \rangle$ denote statistical averaging over stochastic force realisations. Due to the fact that the Dirac function is the Fourier transform of the unity, we have:

$$\delta[x - X(t)] = \int_{-\infty}^{\infty} \frac{dk}{2\pi} exp\{-ik[x - X(t)]\} \tag{4}$$

When averaged, Eq.4 yields merely that the probability is the inverse Fourier transform of the characteristic function $Z_X(k,t)$ (see the Appendix A for an alternative derivation exploiting more directly this property):

$$Z_X(k,t) = \langle exp(ikX(t)]) \rangle \tag{5}$$

$$p(x,t) = F^{-1}[Z_X(k,t)] \tag{6}$$

[1] Following the approach of Einstein and Schmoluwski we neglect the inertial term for large time lags, and therefore consider the balance between viscous friction and forcing.

[2] These pulses correspond to a force divided by the friction coefficient, i.e. the inertial mass divided by the viscous relaxation time.

where F and F^{-1} denote respectively the Fourier–transform and its inverse:

$$F[f] = \hat{f}(k) = \int_{-\infty}^{\infty} dx \;\; exp(ikx) f(x) \quad F^{-1}[\hat{f}] = f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \;\; exp(-ikx)\hat{f}(k) \tag{7}$$

On the other hand, Eq.2 can be integrated into:

$$X(t) = X(0) + \int_0^t d\tau Y(\tau) \tag{8}$$

Since we can assume [3] without loss of generality that $X(0) = 0$, we obtain the following equation:

$$\frac{\partial p}{\partial t} = F^{-1}[\frac{\partial}{\partial t} \left\langle exp \left[ik \int_0^t d\tau Y(\tau) \right] \right\rangle] \tag{9}$$

Now, to make a further step, it is necessary to specify the statistical properties of the stochastic source. We consider the particular example [25] when the source is represented as a sum of independent stochastic "pulses" acting at equally spaced times t_j [4]:

$$Y(t) = \sum_{j=0}^{\infty} Y_{j,\Delta} \Delta \delta(t - t_j) \qquad . \tag{10}$$

where $t_0 = 0, t_{j+1} - t_j = \Delta$ $(j = 0, 1, 2, ...)$ and the pulses $Y_{j,\Delta}$ are independent stochastic variables having stable Lévy distribution $P\{Y_{j,\Delta}\}$ for all j and which has the following characteristic function [3]

$$Z_{Y_{j,\Delta}}(k) = \langle exp(ikY_{j,\Delta}) \rangle = exp\Delta \left\{ i\gamma k - D|k|^{\alpha} \left[1 - i\beta \frac{k}{|k|} \omega(k, \alpha) \right] \right\} \tag{11}$$

where α, β, γ, D are real constants $(0 < \alpha \leq 2, -1 \leq \beta \leq 1, D \geq 0)$ and $\omega(k, \alpha)$ is defined as:

$$\alpha \neq 1 : \quad \omega(k, \alpha) = tan \frac{\pi\alpha}{2}; \qquad \alpha = 1 : \quad \omega(k, \alpha) = \frac{\pi}{2} log|k| \tag{12}$$

α and β classify the type of the stable distributions up to translations and dilatations: with given α and β, γ and D can vary without changing the type of a stable distribution. The parameter α characterizes the asymptotic behaviour of the stable distribution:

$$p(x) \sim x^{-1-\alpha}, x \to \infty \tag{13}$$

[3]Indeed, we are considering only the 'forward' Fokker-Planck equation.

[4]See Sect.6 for an alternative corresponding to a power-law distribution of the waiting times [12, 13]

hence, corresponds to the critical order of moments for their divergence:

$$\mu \geq \alpha : \langle x^\mu \rangle = \infty, \tag{14}$$

For (additive) walks α is also related to the fractal dimension of the trail [10], whereas for the generator of the (multiplicative) universal multifractals it measures their multifractality [19]. The parameter β characterizes the degree of asymmetry of distribution function. Indeed, if $\beta = 0$, then negative and positive values of $Y_{j,\Delta}$ occur with equal probabilities, while if $\beta = 1$ or $\beta = -1$ (maximally asymmetric distributions) then, for $0 < \alpha < 1$ and $\gamma = 0$ $P\{Y_{j,\Delta}\}$ vanishes outside from $[0, +\infty]$ or respectively from $[-\infty, 0]$ [5]. We already mentioned that maximal asymmetry is required for generators of universal multifractals; let us add that in this case the Laplace transform is more convenient than the Fourier transform. The nonzero value of β implies the existence of a primary direction of the stochastic pulses (that is, the direction to plus or minus infinity), and thus the existence of a drift for particles in this direction. For more details concerning the properties of stable laws see, e.g. [26]. The meaning of γ and D will be discussed and clarified below.

Now, using Eq.10 and the independence condition of the stochastic pulses $Y_{j,\Delta}$ we get:

$$\left\langle exp\left[ik\int_0^t d\tau y(\tau)\right]\right\rangle = \left\langle exp\left[ik\sum_{j=0}^{n} Y_{j,\Delta}\right]\right\rangle = \prod_{j=0}^{n}\langle exp\langle ikY_{j,\Delta}\rangle\rangle = \langle exp(ikY_{j,\Delta})\rangle^n \tag{15}$$

where n is a number of pulses corresponding to the present time $t = n\Delta$. Therefore , with the help of the equation of the characteristic function of the pulses (Eq.11), we obtain the characteristic function $Z_X(k,t)$ (Eq.5) of the stable process:

$$Z_X(k,t) = \left\langle exp\left[ik\int_0^t d\tau Y(\tau)\right]\right\rangle = exp\left\{t\left[i\gamma k - D|k|^\alpha\left(1 - i\beta\frac{k}{|k|}\omega(k,\alpha)\right)\right]\right\} \tag{16}$$

The fact that this process has stationary independent increments [27] (i.e. pulses $Y_{j,\Delta}$) gives the possibility to get directly Eq.16 without using any discretisation of $Y(t)$ as previously done (Eq.10). Such a derivation is presented in Appendix A.

Now inserting this expression of $Z_X(k,t)$ into Eq.9, one obtains:

$$\frac{\partial p}{\partial t} = \int_{-\infty}^{\infty}\frac{dk}{2\pi}[i\gamma k - D|k|^\alpha + i\beta D\omega(k,\alpha)k|k|^{\alpha-1}]Z_X(k,t)exp(-ikx) \tag{17}$$

[5] For $\alpha > 1$, $P\{Y_{j,\Delta}\}$ decays faster than an exponential on the corresponding half axis.

For the sake of the simplicity of notations, we will consider in the following only the case $\alpha \neq 1,\ or\ \beta = 0$. Therefore, Eq.12 reduces to:

$$\omega(k, \alpha) \equiv \omega(\alpha) = tan\frac{\pi\alpha}{2} \tag{18}$$

2.2 An expression of the Fractional Fokker–Planck Equation

One can see that in Eq.17 the following type of integrals appears $F^{-1}(|k|^{\alpha} Z_X]$, which in fact correspond to fractional differentiations. Indeed, one may use Laplacian power for the Riesz's definition of a fractional differentiation since for any function $f(x)$:

$$-\Delta f(x) = F^{-1}(|k|^2 \hat{f}(k)) \tag{19}$$

yields a rather straightforward extension:

$$(-\Delta)^{\alpha/2} f(x) = F^{-1}(|k|^{\alpha} \hat{f}(k)) \tag{20}$$

Then, Eq.17 yields:

$$\frac{\partial p}{\partial t} + \gamma\frac{\partial p}{\partial x} = -D\left[(-\Delta)^{\alpha/2} p + \beta\omega(\alpha)\frac{\partial}{\partial x}(-\Delta)^{(\alpha-1)/2} p\right] \tag{21}$$

which for symmetric laws $\beta = 0$ is a straightforward generalization of the classical Fokker–Planck equation, by:

$$\Delta \rightarrow -(-\Delta)^{\alpha/2} \tag{22}$$

This also points out that the scale parameter D of the Lévy distribution corresponds to the diffusion coefficient of the Fractional Fokker Planck equation. On the other hand, the second term in the left hand side of Eq.21 has an obvious physical meaning. Independently on the value of α, it describes the convection of particles by the (constant) velocity γ. For $\alpha > 1,\ \gamma$ corresponds furthermore to the mean value of the source $\langle Y(t)\rangle$, whereas it is no more the case for $\alpha \leq 1$ since the latter is no longer finite. In the latter case, the diffusion term has a a derivation order smaller or equal to the convection term. This confirms that the case $\alpha = 1$ is indeed critical between two rather distinct regimes and it is more involved than other cases. Besides, it is worthwhile to note the role of the term (on the r.h.s.) related to asymmetry ($\beta \neq 0$). On the one hand, this term can be interpreted as an additional contribution to the convection due to existence of the preferred direction of the pulses related to ($\beta \neq 0$). On the other hand, such a flow is not proportional to p (as the convective flow does) but rather to $(-\Delta)^{(\alpha-1)/2}p$, which is rather typical for the diffusion flow. In some sense, due to this term the division of flows into convective and diffusion ones (as done in the standard Fokker–Planck equation) becomes rather questionable and

presumably no longer relevant for the Fractional Fokker–Planck equation. One may note that a somewhat similar weakening of this distinction occurs also in the classical Fokker-Plank for nonlinear systems [28]. On the other hand, it is easy to check that the Fractional Fokker–Planck equation is Galilean invariant, as is should be: the velocity of the moving framework just add to γ.

2.3 The non uniqueness of the expression of the Fractional Fokker-Planck Equation

One cannot expect to obtain a unique expression for the Fractional the Fokker-Planck equation, since there is not a unique generalization of the differentiation to a fractional order. Indeed, there exist various definitions of the fractional differentiation (see, e.g. [29] and references therein) which are not equivalent. This will be illustrated by two examples in the next section. The first one is related to the fact that there are 'signed' (fractional) differentiation and respectively 'unsigned' (fractional) differentiations, i.e. differentiations which are not invariant and respectively invariant with the mirror symmetry $x \to -x$. In the case of standard differentiation, the question of signs is fixed: 'signed' and 'unsigned' differentiations correspond merely to odd and respectively even orders of differentiation (hence the unique expression of the classical Fokker–Planck equation, which is of second order). This is no longer the case for fractional differentiations.

The second example corresponds to the fact that fractional differentiations are in fact defined by integration, and therefore can depend on the bounds of integration.

Nevertheless, we are convinced that the expression corresponding to Eq.21 is at the same time the simplest one to derive and the one whose physical significance is the most straightforward. On the other hand, let us emphasize that the existence of distinct expressions for the Fractional Fokker-Planck equation does not question the uniqueness of its solution. Indeed, these distinct expressions are equivalent because their solution should correspond to the unique probability density function corresponding to a given Langevin–like equation (Eq.2).

The non uniqueness could be rather understood in the following way: corresponding to the distinct fractional differentiations (and their corresponding fractional integrations), there should be distinct ways of solving the Fractional Fokker-Planck equation in order to obtain its unique solution.

2.4 Two alternative expressions of the Fractional Fokker-Planck Equation

Contrary to the unsigned fractional power of a Laplacian Eq.20, let us consider for instance the following 'signed' fractional differentiation:

$$\frac{\partial^\alpha}{\partial x^\alpha} f(x) = F^{-1}[(-ik)^\alpha \hat{f}(k)]. \tag{23}$$

With the help of (i) the identity ($\theta(k)$ being the Heaviside function):

$$|k|^\alpha = k^\alpha[\theta(k) + (-1)^\alpha \theta(-k)] \tag{24}$$

and of (ii) the inverse Fourier transform of the Heaviside function:

$$F^{-1}[\theta(k)] = \frac{1}{2}\delta(x) + \frac{1}{2\pi i x} \tag{25}$$

as well as of (iii) the property that a Fourier transform of a product corresponds to the convolution of the Fourier transforms, one derives from Eq.17 an another form of the Fractional Fokker–Planck equation.

$$\frac{\partial p}{\partial t} + \gamma \frac{\partial p}{\partial x} = -D\left(cos\frac{\pi\alpha}{2} + \beta sin\frac{\pi\alpha}{2} tan\frac{\pi\alpha}{2}\right)\frac{\partial^\alpha p}{\partial x^\alpha} - D(1-\beta) sin\frac{\pi\alpha}{2}\frac{\partial^\alpha}{\partial x^\alpha}\int_{-\infty}^{\infty}\frac{dx'}{\pi}\frac{p(x',t)}{x-x'} \tag{26}$$

Indeed, with the help of the following determinations[6] $(-i)^\alpha = e^{-i\frac{\alpha\pi}{2}}$, $(-1)^\alpha = e^{-i\alpha\pi}$, Eq.24 yields:

$$|k|^\alpha = (-ik)^\alpha[\theta(k)e^{i\frac{\alpha\pi}{2}} + \theta(-k)e^{-i\frac{\alpha\pi}{2}}] \tag{27}$$

and with the help of Eqs.23,25,27, it is rather straightforward to derive Eq.26.

However, Eq.26 is already rather involved in the case $\beta = 0$, whereas this case is obvious for the equivalent Eq.21:

$$\frac{\partial p}{\partial t} = -\gamma\frac{\partial p}{\partial x} - Dcos\frac{\pi\alpha}{2}\frac{\partial^\alpha p}{\partial x^\alpha} - Dsin\frac{\pi\alpha}{2}\frac{\partial^\alpha}{\partial x^\alpha}\int_{-\infty}^{\infty}\frac{dx'}{\pi}\frac{p(x',t)}{x-x'} \tag{28}$$

the last term of the r.h.s. of Eq.28 is rather complex, whereas indispensable. Indeed, there is a need of signed second term to counterbalance the first signed term of Eq. 28, in order that the r.h.s. of Eq.28 will correspond to an unsigned differentiation (the fractional power of the Laplacian in Eq.21). Both terms correspond to the signed fractional differentiation of order α but whereas it is applied to p in the former term, it is applied to an integration of a zero order of p in the latter term. This zero order integration corresponds to the effective interaction of particles having a scaling law inversely proportional to the distance between them. An analogy with the interaction between dislocation lines [30] can be mentioned. It is plausible that the collective effect corresponding to this the effective interaction of particles could be responsible of the large jumps which are so important in Lévy motions.

[6] One may note that the existence of other determinations confirms the non uniqueness of the fractional derivative defined in eq.24. Furthermore, taking another determination will merely modify some prefactors in r.h.s. of Eq.26

An other expression of the Fractional Fokker–Planck equation can be also obtained with the help of the Riemann–Liouville derivatives. The μ-th order Riemann–Liouville derivatives on the real axis are defined as

$$(\mathbf{D}_+^\mu f)(x) = \frac{1}{\Gamma(1-\mu)}\frac{d}{dx}\int_{-\infty}^{x} dx' \frac{f(x')}{(x-x')^\mu} \quad (\mathbf{D}_-^\mu f)(x) = -\frac{1}{\Gamma(1-\mu)}\frac{d}{dx}\int_{x}^{\infty} dx' \frac{f(x')}{(t-x')^\mu} \tag{29}$$

where $\mathbf{D}_-^\mu, \mathbf{D}_+^\mu$ are respectively the left-side and the right-side derivatives of fractional order μ $(0 < \mu < 1)$ and Γ is the Euler's gamma-function. Appendix B gives a derivation of the corresponding expression of the Fractional Fokker–Planck equation which is:

$$\frac{\partial p}{\partial t} + \gamma\frac{\partial p}{\partial x} = -D\mathbf{D}_+^{\alpha/2}\mathbf{D}_-^{\alpha/2}p - D\beta\omega(\alpha)\frac{\partial}{\partial x}\mathbf{D}_+^{(\alpha-1)/2}\mathbf{D}_-^{(\alpha-1)/2}p \tag{30}$$

3 Scaling properties of the Fractional Fokker-Plank equation

Now let us return to one of the equivalent expressions (Eqs.21, 26, 30) of the Fractional Fokker-Plank equation and consider its scaling group properties which defines fundamental properties of its solutions. Without loss of generality we can assume $\gamma = 0$. The scale transformation group can be written in the following manner:

$$t = \lambda t', \quad x = \lambda^\chi x', \quad D = \lambda^\kappa D' \tag{31}$$

$$p(x,t;\alpha,\beta,D) = \lambda^\delta p((x',t';\alpha,\beta,D') \tag{32}$$

Here χ, κ, δ are yet unknown exponents of the scale transformations which should leave invariant the Eqs.21, 26, 30. One may note that the subgroup not dealing with D transformations was used by [12] in order to obtain the fractional order of differentiation of the Fractional Fokker-Planck equation corresponding to a symmetric stable processes (see Sect.6). Due to the further normalization condition for the distribution function, one obtains the following 2-parameters scale transformation group:

$$t = \lambda t', \quad x = \lambda^\chi x', \quad D = \lambda^{\alpha\chi-1} D' \quad . \tag{33}$$

$$p(\lambda^\chi x, \lambda t; \alpha, \beta, \lambda^{\alpha\chi-1}D) = \lambda^{-\chi} p(x,t;\alpha,\beta,D) \quad . \tag{34}$$

χ, λ being the arbitrary group parameters.

If furthermore the initial condition $p(x,0)$ is invariant under the scaling group (Eq.33), then the corresponding solution of Eqs.21, 26, 30 remains invariant

under the action of this group for any other time. The simplest example of an invariant initial condition is $p(x,0) = \delta(x)$, which is of fundamental importance since it corresponds to the Green functions of Eqs.21, 26, 30.

Let us analyze the general properties of the invariant solutions in a similar way to renormalization group approach [31, 32] (analogous consideration was used in a more complex variant when calculating the spectrum of a compressible fluid [33]). Due to the fact that the scaling invariant solutions should satisfy the identity (Eq.34) for any value of the arbitrary parameters λ, χ, they could depend only on products of variables which are independent of them. Therefore, due to the relationships $\alpha\chi - \kappa - 1 = 0$ and $\delta + \kappa = 0$, the scaling solutions are:

$$p(x,t) = \frac{1}{x}\Phi\left(\frac{x^\alpha}{Dt}\right) \equiv \frac{1}{(Dt)^{\frac{1}{\alpha}}}\Psi\left(\frac{x^\alpha}{Dt}\right) \tag{35}$$

where $\Phi(.) = p(1,.), \Psi = p(.,1)$ are arbitrary functions which are determined by the initial conditions. Therefore, Eq.35 represents the general form of the invariant solutions of the kinetic equation Eqs.21, 26, 30.

Eq. 35 can be obtained by first differentiating Eq.34 with respect to λ, and then, to χ and setting $\lambda = 1, \chi = -1/\alpha$. This yields the following system of equations:

$$t\frac{\partial p}{\partial t} + \frac{x}{\alpha}\frac{\partial p}{\partial x} = -\frac{1}{\alpha}p, \qquad x\frac{\partial p}{\partial t} + D\alpha\frac{\partial p}{\partial D} = -p \tag{36}$$

which are linear and therefore can be solved by the method of characteristics and their solution indeed correspond to Eq.35.

4 Some particular solutions

4.1 Explicit solutions

With the exception of the three following cases, there is no way to obtain an explicit expression of the solutions, with the initial condition $f(x,0) = \delta(x)$, of the Fractional Fokker-Plank equation (Eqs.21, 26, 30 in a closed form with the help of elementary functions:

1) $\alpha = 2$ ($\beta = 0$):

It corresponds to Gaussian distribution of the stochastic forcing and to the classical Fokker–Planck equation:

$$\frac{\partial p}{\partial t} + \gamma\frac{\partial p}{\partial x} = D\frac{\partial^2 p}{\partial x^2} \quad . \tag{37}$$

which solution is the normal distribution:

$$p(x,t) = \frac{1}{\sqrt{4\pi Dt}} exp\left[-\frac{(x-\gamma t)^2}{4Dt}\right] \tag{38}$$

2) $\alpha = 1, \beta = 0$:
It corresponds to a forcing having a Cauchy distribution and to the following Fractional Fokker-Planck equation:

$$\frac{\partial p}{\partial t} + \gamma \frac{\partial p}{\partial x} = D \frac{\partial}{\partial x} \int_{-\infty}^{\infty} \frac{dx'}{\pi} \frac{p(x',t)}{x'-x}. \tag{39}$$

The solution of Eq.39 with the initial condition $f(x,0) = \delta(x)$ is:

$$p(x,t) = \frac{Dt}{\pi} \frac{1}{(x-\gamma t)^2 + D^2 t^2}, \tag{40}$$

3) $\alpha = 1/2, \beta = 1$.
Then the expression displayed in Eq.26 of the Fractional Fokker-Planck equation takes then the form:

$$\frac{\partial p}{\partial t} + \gamma \frac{\partial p}{\partial x} = \sqrt{2} D \frac{\partial^{1/2} p}{\partial x^{1/2}}. \tag{41}$$

The solution of Eq.41 with the initial condition $f(x,0) = \delta(x)$ is:

$$p(x,t) = \theta(x-\gamma t) \frac{Dt}{\sqrt{2\pi (x-\gamma t)^{3/2}}} exp\left[-\frac{D^2 t^2}{2(x-\gamma t)} \right], \tag{42}$$

It is easy to check that all the explicit stable distributions (Eqs.38, 40, 42) belong to the class of scale invariant solutions.

5 Generalization of Einstein relation and anomalous diffusion coefficient

The scaling analysis of the moments of the distribution function will lead to the generalization of the Einstein relation for the anomalous diffusion. However, there is an important difference, since moments of order larger than $\alpha < 2$ will diverge and in particular: $\langle x^2 \rangle = \infty$. On the other hand, the motion remains mono-fractal, since all the moments $\langle x^\mu \rangle$ $(0 < \mu < \alpha)$ will have the same scaling law.

The statistical moments of the distribution function of particles initially concentrated at the origin ($p(x,0) = \delta(x)$) correspond to :

$$\langle x^\mu \rangle = \int_{-\infty}^{\infty} dx x^\mu \int_{-\infty}^{\infty} \frac{dk}{2\pi} exp\left[-ikx - D|k|^\alpha t \left(a - i\beta \frac{k}{|k|} tan \frac{\pi\alpha}{2} \right) \right], \tag{43}$$

In agreement with the scaling properties obtained in Sect.3), we have:

$$r_\mu(t) \equiv \langle x^\mu \rangle^{1/\mu} = (\alpha D t)^{1/\alpha} C(\alpha, \beta, \mu) \tag{44}$$

which is obtained by renormalizing x and k by $(Dt)^{1/\alpha}$, i.e. by considering the following variables:

$$x_1 = k(\alpha Dt)^{1/\alpha}, x_2 = \frac{x}{(\alpha Dt)^{1/\alpha}} \tag{45}$$

which yield from Eq.43 the following prefactor (depending neither on time nor on D) :

$$C(\alpha,\beta,\mu) = \left\{\int_{-\infty}^{\infty} dx_2 x_2^{\mu} \int_{-\infty}^{\infty} \frac{dx_1}{2\pi} exp\left[-ix_1x_2 - (\frac{|x_1|^\alpha}{\alpha}\left(1 - i\beta sgn(x_1) tan(\frac{\pi\alpha}{2})\right)\right]\right\}^{1/\mu} \tag{46}$$

It follows from Eq.44 that the scaling of $r_\mu(t)$ in respect to D and t is universal and does not depend on the order μ of the moment considered ($\mu < \alpha$). Therefore the Einstein relation can be formulated in terms of any of the finite moments. Indeed, only the the numerical prefactor $C(\alpha,\beta,\mu)$ (Eq.46) depends on μ, but neither on time nor on D.

The Gaussian case yields the classical Einstein formula :

$$r_2(t) = (2Dt)^{1/2} \tag{47}$$

Not surprisingly, Eq.44 confirms the fact, already pointed out on Eq.21, that D does corresponds to a (generalized) diffusion coefficient.

On the other hand, let us confirm that the scaling behaviour (Eq.44) is independent of the initial distribution of the particles. This is due to the fact that the distribution with initial condition $f(x,0) = \delta(x)$ plays the role of the Green function for the distributions with other conditions, and therefore imposes its scaling on time and on D. This is easily confirmed with the help of Eq.35 which gives the general expression of the scaling probability densities:

$$r_\mu(t) = (Dt)^{1/\alpha}\tilde{C}(\alpha,\beta,\mu)$$

with:

$$\tilde{C}(\alpha,\beta,\mu) = \left\{\int_{-\infty}^{\infty} d\xi \xi^{\mu-1}\Phi(\xi^\alpha)\right\}^{1/\mu} \tag{48}$$

6 Comparison with other approaches

As mentioned in Sect.1, particular cases of the Fractional Fokker-Planck equation were obtained [12, 13, 14], and on the other hand a Langevin-type equation has been obtained[17] for the nonlinear Fokker-Planck equation [15, 16] whose solutions maximize the generalized q-entropy introduced by [18] and exhibit a Levy-like anomalous scaling.

A Fractional Fokker-Planck equation was obtained by [12, 13] in the framework of the continuous time random walks (CTRW's) model of anomalous diffusion [35]. However, this method does not involve directly a Lévy process, but a walk sharing some common behaviour of the latter, without being equivalent to it. Indeed, the distribution of steps, which corresponds to the probability distribution of the pulses in the Langevin equation, is considered as a pure power-law. This corresponds only to the asymptotic behaviour of the Lévy distribution, i.e. its tails, and therefore takes into account only one of the Lévy law parameters. This nevertheless allows to establish scaling relations, therefore to determine the fractional order of differentiation (see however a remark below), but not to determine a precise expression of this fractional differentiation. This is already the case for its coefficient, i.e. the (fractional) diffusion coefficient of the Fokker-Planck equation, since scaling reasoning does not yield a relation with the scale parameter. Second, there is no simple way to deal with the skewness parameter β when considering the probability distribution. Therefore, the corresponding non trivial term in the Fokker-Planck (Eqs.21, 26, 30) was not obtained by [12, 13]. On the contrary, in our approach the four parameters, which determine all the statistics of the pulses, are all taken into account in an exact and rather straightforward manner with the help of the characteristic function. On the other hand, [12, 13] show an easy generalization to both temporal and spatial memories is obtained by introduced a a second (generalized) Langevin equation for the waiting times, which introduces a Fractional Fokker-Planck equation for the latter. However, let us point out that the usual scaling reasoning does not apply in a straightforward manner due to the divergence of moments associated to the power-law of the Lévy probability distribution. Indeed, one cannot consider the scaling of the variance of the distance $\langle r^2(s) \rangle$ traveled by a particle after s steps, because it is a divergent statistical moment, i.e. equals to infinity, as soon as $\alpha \neq 2$. One has to consider moments of order $\mu < \alpha$, which are not only finite but are furthermore monoscaling (Sect. 5). This last property means that the scaling of the $\mu - th$ root of moment of order μ is independent of this order, and therefore explains why the (mono-) scaling reasoning works.

In a recent article [14] a different form of fractional Fokker-Planck was introduced with the help of a phenomenological and interesting modification of the classical Fick law into a fractional Fick law, which is discussed in details. The question of the existence of a corresponding Langevin–like equation remains open. Some integral convergence problem imposes that the power-law exponent of the probability distribution tails belongs to $]1, 2]$. This exponent would correspond to the Lévy stability index α the solution of this fractional Fokker-Planck was a Lévy motion. However, this is not exactly the case, although it seems at first glance rather similar to it. Indeed the corresponding characteristic function (see Eqs. 14-15 of Ref.[14] involves k^α instead of $(ik)^\alpha$ for the characteristic function of a Lévy motion (Eq. 11). The difference is more obvious when considering the asymmetric extension, either on the proposed Fokker-Planck equation (Eq.

17 of Ref.[14] or on the characteristic function (Eq. 19 of Ref.[14]), since it does not include the non trivial asymmetric term that we put in evidence (Eqs.21, 26, 30). Nevertheless, it corresponds to an interesting variant of asymmetric diffusion, which solutions could be rather close to stable Levy distributions.

An essentially different generalization of the Fokker-Planck equation was obtained by [16, 17] for anomalous diffusion. Indeed, [16] showed that the solutions of the nonlinear Fokker-Planck equation introduced by [15, 16], maximize the generalized q-entropy [18], and correspond to a well defined Lévy-like anomalous diffusion, but with a finite variance and non zero correlation. Both properties, which seem relevant and desirable for many applications, in particular for the transport in porous media, are not satisfied by a Lévy motion.

Furthermore, [17] demonstrated that the corresponding Langevin-like equation has the particularity that the random forces are modulated by a given power of the probability distribution. This corresponds to a macroscopic feedback to the microscopic kinetics, which is absent in our Langevin equation.

In comparison with these different works, we followed a rather distinct approach since we started with a Langevin-like equation with random forces which are *exact* stable Levy processes, which can be symmetric as well as asymmetric, and with no limitation on the possible values of the Levy index α. The particular case corresponding the symmetric stable processes was previously inferred by [12, 13, 14]. However, we showed that in the more general case of asymmetric stable processes, a new non-trivial term appears which has a rather intermediate role between diffusion and convection (see Sect.2.2). Furthermore, the use of the characteristic function allowed us to obtain a generalization of Einstein's formula. We also clarify the fact that different expressions of the same Fractional Fokker-Planck equation are obtained, depending on the type of fractional differentiation which is used.

On the other hand, the conjecture issued by [16] that 'further unification can be possibly achieved by considering the generic case of a *nonlinear* Fokker-Planck -like equation with *fractional* derivatives' should be closely examined, as well as the fractional time evolution suggested by Fogedby, Compte.

7 Examples of applications

7.1 Generalisation of the Holtzmark distribution

In the introduction of this paper, we recalled that the gravitational force resulting from randomly and homogeneously distributed point masses which acts on a given test point mass has [6, 7] has a stable symmetrical law with $\alpha = 3/2$. One may note that a similar problem arises with charged particles and electrostatic forces. However, the distribution of masses in the Universe is rather inhomogeneously distributed, e.g. on a fractal set of fractal dimension [36] $D_F \approx 1.2$. (see [37] for discussion and a multifractal analysis). Let us extend the original result

of Holtzmark to the case of particles distributed on a (possibly fractal) space of dimension d and which mutually interact according to a scaling law $1/L^r$, L being the distance between two particles, e.g. the collective force $\mathbf{f}$ acting on the randomly chosen particle of unit mass located at $\mathbf{x}$, which results from the distribution of masses m_k at points $\mathbf{x}^{(k)}$, has the following type [7] :

$$\mathbf{f}(\mathbf{x}) = \sum_k m_k \frac{\mathbf{x}^{(k)} - \mathbf{x}}{|\mathbf{x} - \mathbf{x}^{(k)}|^{r+1}} \tag{49}$$

Due to the linearity of Eq.49, the superposition ($[m+m']$) of two independent mass distributions ($[m]$ and $[m']$) yields a force having the same probability distribution as the one of the sum of the two forces resulting from each of mass distribution, i.e.:

$$\mathbf{f}([m]) + \mathbf{f}([m']) \overset{\mathrm{d}}{=} \mathbf{f}([m+m']) \tag{50}$$

on the other hand, the fact that the mass is concentrated on a fractal set of dimension D_F, it should scale in the following manner with the (space) scale resolution λ, i.e. the ration $\lambda = \frac{L}{l}$ of the outer scale L over the inner scale l of the fractal set, in particular in the limit $\lambda \rightarrow \infty$:

$$m_\lambda = m_1 \lambda^{-D_F} \tag{51}$$

which implies with the help of Eq.49 the following scaling for the forces:

$$f_\lambda[m_l] \overset{\mathrm{d}}{=} \lambda^{-r} f_1[m_l] \overset{\mathrm{d}}{=} \left(\frac{m_l}{m_1}\right)^{r/D_F} f_1[m_l] \tag{52}$$

which together with Eq.50 demonstrates [4] that $f_\lambda[m_\lambda]$ has a (symmetric) Lévy stable distribution, with a Lévy index $\alpha = D_F/r$.

With this Lévy index value for the random forces, and neglecting their inter-relations (which will be studied elsewhere), we may define the random velocity of the test mass as defined by Eq.2, and its probability distribution by Eq.21.

7.2 The anomalous diffusion of a passive scalar by a two-dimensional turbulence

Let us consider the velocity field $v_i(\mathbf{x})$ resulting from point-like vortices:

$$v_i(\mathbf{x}) = \sum_k \frac{\kappa}{2\pi} \frac{\varepsilon_{ij}(x_j - x_j^{(k)})}{|\mathbf{x} - \mathbf{x}^{(k)}|^2} \tag{53}$$

where κ is the intensity of the vortices, $\varepsilon_{ij}; ij = 1,2$ is the fundamental anti-symmetric tensor, $\mathbf{x}^k$ is the location of the k^th vortex. Following [38] we assume

[7] by 'type', we mean that most of the algebraic details of the following equation are irrelevant, only its scaling properties are relevant.

that the vortices are distributed on a fractal set of fractal dimension D_F. We are therefore in the situation of the generalization of Holtzmark distribution and indeed Eq. 53 is of the type of Eq. 49 with $r = 1$. Therefore, $\mathbf{v}[(\kappa)]$ has a Lévy probability distribution with $\alpha = D_F$. one may note that this result can be obtained by using straightforward calculations of the characteristic function in the manner analogous to [7]. The main distinction is that the fractal distribution of the vortices had be taken into account.

Therefore, the diffusion of a passive scalar in $2D$ turbulence created by a fractal set of point–like vortices is defined by Eq.21 with $\gamma = \beta = 0$ and $\alpha = D_F$. It points out the interest of using Fractional Fokker–Planck equation for the analysis of the diffusion processes of particles in turbulent media.

7.3 Multifractal modeling

However, as noted in the introduction, the most appealing area of application of the Fractional Fokker–Planck equation could be for $3D$ turbulence. Indeed it should play a key role for the definition of the generators of *dynamic* universal multifractals [39]. Let us first recall some basic features of *static* universal multifractals, i.e. defined only on space. The corresponding field, e.g. the flux of energy F_λ at higher and higher resolution $\lambda = \frac{L}{l}$, should respect the multiplicative property of the scale ratio, i.e.:

$$F_{\Lambda=\lambda\cdot\lambda'} = F_\lambda \cdot T_\lambda(F_{\lambda'}) \tag{54}$$

where T_λ is a scale contraction operator of ratio λ, which in the simplest case is the isotropic self-similar contraction ($T_\lambda(\mathbf{x}) = \lambda^{-1}\mathbf{x}$). Therefore F_Λ might be defined with the help of the generator Γ of this group , more precisely speaking by the exponentiation of the latter which satisfies the following additive property:

$$\Gamma_{\Lambda=\lambda\cdot\lambda'} = \Gamma_\lambda + T_\lambda(\Gamma_{\lambda'}) \tag{55}$$

In order to satisfy the multiscaling power law:

$$\forall\lambda \in (1, \Lambda) : \ \langle F_\lambda^q \rangle \sim \lambda^{K(q)} \tag{56}$$

the generator should have a logarithmic scale divergence:

$$\Gamma_\lambda \sim \log \ \lambda \tag{57}$$

this latter condition is obtained by a convolution of a given the Green's function g over a white-noise γ_λ (called the 'sub-generator'):

$$\Gamma_\lambda = g \star \gamma_\lambda \tag{58}$$

In the case of universal multifractals [19] the sub-generator is an extremely asymmetric and centered Lévy stable with a Lévy index α and the condition of

logarithmic of divergence (for a $D-$dimensional isotropic process) corresponds to:

$$g^{\alpha}(\underline{x}) \propto \mid \underline{x} \mid^{-\alpha.D_H}; D_H = \frac{D}{\alpha} \tag{59}$$

However, in order to take into account the causality for time-space processes [24], it is rather more interesting to consider a differentiation operator, i.e. to consider g^{-1} rather than g, i.e.:

$$g^{-1}(\underline{x}, t) \star \Gamma_{\Lambda}(\underline{x}, t) = \gamma_{\Lambda}(\underline{x}, t) \tag{60}$$

Furthermore, in order to take into account the difference of scaling in space and time, the time and respectively space orders of differentiation should be different. Therefore, the following type of differential equation were considered:

$$g^{-\frac{\alpha}{D_{el}}} = \frac{\partial}{\partial t} + (-\Delta)^{1-H_t} \tag{61}$$

where the 'elliptical dimension of the space' $D_{el} = D + 1 - H_t$ is the effective space-time dimension, D is the dimension of the space cut and H_t corresponds to the deviation of the time scaling in comparison to the time scaling. The Fractional Fokker-Plank equation (Eq.21) suggests that the following fractional operators could be as well considered:

$$g^{-\frac{\alpha}{D_{el}}} = \frac{\partial}{\partial t} + (-\Delta)^{1-H_t} + \beta\omega(\alpha)\frac{\partial}{\partial \mathbf{x}}(-\Delta)^{(1-H_t-1)/2} \tag{62}$$

and for any value of β the evolution of the generator has a microscopic interpretation with the help of the corresponding Langevin equation, i.e. it points out that Eq.58 corresponds to a (generalized) path integral.

8 Conclusions

The original results obtained in this paper are the following:

1. the Fractional Fokker–Planck equation, i.e. the kinetic equation describing anomalous diffusion in response to a stochastic forcing having a Lévy stable distribution, which can be symmetric, as well as asymmetric,

2. the a physical interpretation of all the parameters of the Lévy stable distributions, due to a precise determination of the coefficients of the Fractional Fokker–Planck equation,

3. the scale transformation group of the Fractional Fokker–Planck equation, as well as corresponding scaling solutions,

4. the universal dependence of the distribution function moments on the diffusion coefficient and time,

5. some preliminary examples of applications, including a generalisation of the Holtzmark distribution, two-dimensional diffusion of a passive scalar and multifractal modeling of intermittent fields.

We compare these results with particular cases obtained by [12, 13, 14], as well as their relations to a nonlinear Fokker-Planck equation introduced by [15, 16] whose solutions exhibit an anomalous Lévy-like diffusion [16, 17].

In summary, we believe that the kinetic equation obtained will be useful for studying various physical systems with non-Gaussian statistics.

9 Acknowledgments

This paper was supported by International Association under the project INTAS-93-1194 and by the State Committee on Science and Technologies of Ukraine, grant No 2/278.]. We acknowledge stimulating discussions with Academician S.V.Peletminsky and Prs. J. Brannan, J. Duan, Yu.L.Klimontovich, M. Larcheveque and S. Lovejoy. We thank Pr. N.G. Van Kampen for his very careful reading of earlier versions of this paper. We are grateful to two anonymous referees for their helpful comments and stimulating suggestions.

A Another Way to Derive the Fractional Fokker-Plank Equation.

Here we present another approach to the derivation of Eq.16. Let be the probability distribution function $p(x,t)$ of $x(t)$ which obeys Eq.2 and having first characteristic function $Z_X(k,t)$:

$$p(x,t) = F^{-1}[Z_X(k,t)] \tag{A.1}$$

Due to the linearity of the Fourier transform and the fact that the one considered applies only in space, not in time:

$$\frac{\partial p}{\partial t} = F^{-1}\left(\frac{\partial Z_X(k,t)}{\partial t}\right) \tag{A.2}$$

If the pulses $Y_{dt}(t)'s$ for infinitesimal time lag dt are independent and identically distributed variables for any arbitrary time t, and have a second characteristic function $dtK_Y(k)$:

$$Z_{Y\,dt}(k,t) = exp[idtK_Y(k)] \tag{A.3}$$

then $X(t)$ has independent increments [27] and:

$$Z_X(k,t) = exp[itK_Y(k)] \tag{A.4}$$

The demonstration of Eq. A.4 is rather straightforward, but can be also obtained with the help of the time discretisation which we used in order to obtain Eq.15.

Inserting Eq.(A.4) into Eq.(A.2) we get

$$\frac{\partial p}{\partial t} = F^{-1}[iK_y(k)Z_X(k,t)] \tag{A.5}$$

The particular case of Lévy stable pulses $Y(t)$ corresponds to

$$K_Y(k) = \gamma k + iD|k|^\alpha \left[1 - i\beta \frac{k}{|k|}\omega(k,\alpha)\right] \tag{A.6}$$

therefore, we immediately get Eq.17, and we only need to interpret $|k|^\alpha$ and $k|k|^{\alpha-1}$ in the physical space, as done in Sect.2.

B Fractional Fokker–Planck Equation in Terms of Riemann–Liouville Derivatives.

The μ-th order Riemann–Liouville derivatives on the infinite axis are defined as

$$(\mathbf{D}^\mu_+)(x) = \frac{1}{\Gamma(1-\mu)} \frac{d}{dx} \int_{-\infty}^{x} dt \frac{f(t)}{(x-t)^\mu} \tag{B.1}$$

$$(\mathbf{D}^\mu_-)(x) = \frac{1}{\Gamma(1-\mu)} \frac{d}{dx} \int_{x}^{\infty} dt \frac{f(t)}{(t-x)^\mu} \tag{B.2}$$

where $\mathbf{D}^\mu_+, \mathbf{D}^\mu_-$ are respectively the left–side and the right–side derivatives of fractional order μ $(0 < \mu < 1), \Gamma)$ is the Euler's gamma–function.

The Fourier–transforms of the fractional derivatives (B.1), (B.2) are the following:

$$F(\mathbf{D}^\mu_\pm f) = (\mp ik)^\mu \hat{f}(k) \tag{B.3}$$

where $\hat{f}(k)$ is a Fourier transform of $f(x)$, and we have:

$$(\mp ik)^\mu = |k|^\mu exp\left(\mp \frac{\mu\pi i}{2} signk\right) \tag{B.4}$$

where μ is real and which yields:

$$\mathbf{D}_{+}^{\mu}\mathbf{D}_{-}^{\mu}f = F^{-1}\left[(-ik)^{\mu}(+ik)^{\mu}\hat{f}_k\right] = F^{-1}\left[k^{2\mu}\hat{f}_k\right] = F^{-1}\left[|k|^{2\mu}\hat{f}_k\right] \tag{B.5}$$

therefore

$$\mathbf{D}_{+}^{\alpha/2}\mathbf{D}_{-}^{\alpha/2}f = F^{-1}\left[|k|^{\alpha}\hat{f}_k\right] \tag{B.6}$$

i.e.:

$$\mathbf{D}_{+}^{\alpha/2}\mathbf{D}_{-}^{\alpha/2}f = (-\Delta)^{\alpha/2} \tag{B.7}$$

Eq. B.7 establishes the equivalence between Eq.21 and Eq.30.

References

[1] van Kampen N.G., Stochastic Processes in Physics and Chemistry. North–Holland Physics Publishing (1981).

[2] Khintchine A.Y. and Lévy P., C. R. Acad. Sci.(Paris) **202**, 374 (1936).

[3] Lévy P., Theorie de l"Addition des Variables Aléatoires. Gauthier–Villars, Paris (1937).

[4] Feller W., An Introduction to Probability Theory and its Applications. John.Wiley & Sons, New.York (1966).

[5] Zolotarev V.M., One-dimensional Stable Distributions, MM, Vol. 65, AMS (1986).

[6] Holtzmark J., Ann.d.Physic **58**,162 (1919) ; Phys.Zeits. **20**, 162 (1919); **25**, 73 (1924).

[7] Chandrasekar S., Review of Modern Physics **15**, 1-89 (1943).

[8] Bouchaud J.-P. and Georges A., Phys.Rep. **195**, 127-293 (1990).

[9] Coleman P.H., Fractals ' Physical Origin and Properties, Ed. L. Pietronero, Plenum Press, New York (1989).

[10] Mandelbrot B.B., The Fractal Geometry of Nature. Freeman, San Francisco (1982).

[11] G.M. Zaslavsky, *Physica D, 76*, 110 (1994)

[12] Fogedby,H.C., *Phys.Rev.E, 50*, 2, 1657-1660, (1994)

[13] Compte, A *Phys.Rev.E, 53*, 4, 4191-4193, (1996)

[14] Chaves, A.S., *Phys. Letters A, 239*, 13-16 (1998)

[15] Plastino, A.R., A. Plastino, *Physica A, 222*, 347 (1995)

[16] Tsallis, C., D.J. Bukman,*Phys. Rev. E, 54*, R2197 (1996)

[17] Boland, L., *Phys. Rev. E, 57*, 6, 6634-6642 (1998)

[18] Tsallis, C.,*J. Stat. Phys., 52*, 479 (1988)

[19] Schertzer, D., S. Lovejoy, J. Geophys. Res. **92**, D8, 9693 (1987). Schertzer, D., S. Lovejoy, J. Appl. Meteor. **36**. 1296-1303 (1997).

[20] Wilson, J. , S. Lovejoy, D. Schertzer, Nonlinear Variability in Geophysics: Scaling and Fractals, Eds. D. Schertzer and S. Lovejoy, Kluver Academic Press,Dordrecht-Boston (1991).

[21] Pecknold, S., S. Lovejoy, D. Schertzer, C. Hooge, J.F. Malouin, Cellular Automata: prospects in astronomy and astrophysics, Eds. J.M. Perdang, A. Lejeune, World Scientific, Singapor (1993).

[22] Lavallee, D., S. Lovejoy, D. Schertzer, F. Schmitt, Topological aspects of the dynamics of fluids and plasmas, Eds. K. Moffat, M. Tabor, G. Zaslavsky. Kluwer, Dordrecht (1992).

[23] Schmitt, F., D. Lavallee, D. Schertzer, S. Lovejoy, Phys. Rev. Lett., **68**, 305 (1992).

[24] Marsan, D., D. Schertzer, S. Lovejoy, J. Geophys. Res., **31D**, 26333-26346 (1996).

[25] Chechkin, A.V., D. Schertzer, , A.V. Tur, V.V. Yanovsky, Ukr. J Phys., **40**, 5, 434-439 (1995).

[26] Gnedenko B.V. and Kolmogorov A.N., Limit Distributions for Sums of Independent Random Variables, Gostekhizdat, Moscow–Leningrad, (1949) (in Russian); English translation: Addison Wesley, Reading, MA, (1954).

[27] Skorohod A.V., Stochastic Processes with Independent Increases, Nauka, Moscow, (1964) (in Russian); Studies in the Theory of Random Processes, Addison-Wesley, Reading Mass., 199 pp. (1995).

[28] van Kampen N.G., private communication (1997).

[29] Miller, K.S., An Introduction to the Fractional Calculus and Fractional Differential Equations, J. Wiley & sons, New York (1993).

[30] Landau L.D. and Lifshitz E.M., Theory of Elasticity, Pergamon Press, London (1970).

[31] Stueckelberg E.C.G. and Peterman A., Helv.Phys.Acta, **26**, 499 (1953).

[32] Shirkov D.V., Doklady Acad.Nauk SSSR (Sov.Phys.Dokl.) **263**, 64 (1982).

[33] Moiseev S.S., Tur A.V. and Yanovsky V.V., Sov.Phys.JETP **71**, 1062 (1976).

[34] Klimontovich Yu.L., Statistical Physics. Harwood Academic Publishers, New York (1986).

[35] Klafter, J., A..Blumen,andM.F.Shlesinger, *Phys.Rev.A 35*, 3081 (1987)

[36] de Vaucouleurs G., Science **167**, 1203 (1970).

[37] Garrido, P. , S. Lovejoy, D. Schertzer, Physica A, **225**, 3-4 (1996).

[38] Viecelli J.A., Phys.Fluids A **2**, 2036 (1990).

[39] D. Schertzer, S. Lovejoy, F. Schmitt, Y. Chigirinskaya, D. Marsan, Fractals **5**, 3, 427-471 (1997).